DETERMINATION OF THE QUENCHING STRESSES IN PRISMATIC SPECIMENS BY THE METHOD OF INTEGRAL PHOTOELASTICITY

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UDC 548.552.24

It is assumed that the specimen is made of glass and that the residual stresses are due to quenching, i.e., the plastic-strain tensor is spherical and is described by the effective temperature T [1-3]. The temperature of the specimen is below the glass point, so that Wertheim's integral law [4, 5] is satisfied. The parameters which characterize the specimen do not depend on the axial coordinate z or the components of the stress tensor $\sigma_{xz} = \sigma_{yz} = 0$. As simplifications, we assume that the refractive index and the optical photoelastic constant C do not vary. The specimen is examined in the plane normal to the axis of the specimen z.

The problem being studied will actually be broken down into two parts: 1) determination of σ_{zz} by the method of integral photoelasticity; 2) determination of the remaining components of the stress tensor σ_{ij} using the solution of the first problem and equations of the theory of elasticity. A complete solution to this problem has been obtained only for circular cylinders [6-8]. It was based on the solution of the axisymmetric problem of thermoelasticity for a circular cylinder [9].

In the present study, we extend the method to a section of arbitrary form for an arbitrary distribution of the stress σ_{zz} along the section. The method can be used in particular to determine the quenching stresses created in semifinished multilayered light guides.

Previous studies conducted in this area [5, 10] were limited to finding the stress σ_{zz}.

1. When the specimen is examined in the plane normal to the z axis, only ray integrals are determined [4, 8]

$$C \int (\sigma_{zz} - \sigma_{nn})\, dl = C \int \sigma_{zz}\, dl. \tag{1.1}$$

Here, σ_{nn} is the stress component which is normal to the ray ℓ in the plane x, y. The last equation in (1.1) was obtained from the condition of equilibrium of a segment of the cross section of the prism in the direction n with allowance for the fact that the prism's lateral surface is free of loads and $\sigma_{xz} = \sigma_{yz} = 0$ [10, 11]. Thus, determination of σ_{zz} reduces to the standard procedure of inversion of the Radon transform [11, 12].

To find the remaining components of the stress tensor, we use the equations of equilibrium and Hooke's law for a medium with plastic strains. Meanwhile, following [1-3, 8], we will determine the plastic strains through the effective temperature T by means of the coefficient of linear expansion α: $\varepsilon^0_{xx} = \varepsilon^0_{yy} = \varepsilon^0_{zz} = \alpha T(x, y)$.

In the case of plane strain, Hooke's law in this notation becomes the Duhamel-Neumann relations [13] (where ν is the Poisson's ratio)

$$\sigma_{ij} = 2\mu[\varepsilon_{ij} + \delta_{ij}[\nu e - (1+\nu)\alpha T]/(1-2\nu)],$$
$$\sigma_{zz} = 2\mu[\nu e - (1+\nu)\alpha T]/(1-2\nu),\ e = \varepsilon_{xx} + \varepsilon_{yy},\ i, j = x, y$$

(μ is the shear modulus; δ_{ij} is the Kronecker symbol).

We satisfy the equilibrium equations by introducing the Airy function F:

$$\sigma_{ij} = \delta_{ij}\Delta F - \frac{\partial^2}{\partial i\, \partial j} F. \tag{1.2}$$

Tallinn. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 179-182, July-August, 1991. Original article submitted January 8, 1990; revision submitted March 14, 1990.

Inserting into the compatibility equation $\frac{\partial^2}{\partial y^2}\varepsilon_{xx}+\frac{\partial^2}{\partial x^2}\varepsilon_{yy}=2\frac{\partial^2}{\partial x\,\partial y}\varepsilon_{xy}$ the strain-tensor components $\varepsilon_{ij}=\frac{1}{2\mu}[\sigma_{ij}-\sigma_{zz}\delta_{ij}]$ expressed in terms of F, we obtain the resolvent equation

$$\frac{\partial^2}{\partial x^2}\left[\frac{1}{\mu}\frac{\partial^2}{\partial x^2}F\right]+2\frac{\partial^2}{\partial x\,\partial y}\left[\frac{1}{\mu}\frac{\partial^2}{\partial x\,\partial y}F\right]+\frac{\partial^2}{\partial y^2}\left[\frac{1}{\mu}\frac{\partial^2}{\partial y^2}F\right]=\Delta\left(\frac{1}{\mu}\sigma_{zz}\right). \tag{1.3}$$

The value of F and its normal derivative on the free lateral surface are equal to zero [13].

2. Let us examine certain features of the use of these equations to determine the quenching stresses in semifinished multilayered light guides. Typical of this case is the use of a combination of materials in which the shear coefficient decreases by 2-3%. Meanwhile, the Poisson's ratio may vary by 10-20% [14]. Thus, the difference in the coefficient μ will be no greater than 3% for light guides with a core of borosilicate glass and sheath of quartz glass. We can therefore assume that the coefficient μ is constant for such specimens.

Equation (1.3) can be simplified as follows

$$\Delta^2 F=\Delta\sigma_{zz}. \tag{2.1}$$

The order of Eq. (2.1) can be reduced by writing it in the form

$$\Delta F=\sigma_{zz}-\chi \tag{2.2}$$

(χ is an arbitrary harmonic function).

We will prove that in order to solve the boundary-value problem for F, it is necessary and sufficient that the function χ be equal to the harmonic part of σ_{zz}. For this, we multiply both sides of Eq. (2.2) by an arbitrary, twice-differentiable function u and we integrate the expression over the cross-sectional area

$$\int\int u\Delta F\,ds=\int\int u\,(\sigma_{zz}-\chi)\,ds. \tag{2.3}$$

We transform the left side of Eq. (2.3), using Green's formula

$$\int\int(u\Delta F-F\Delta u)\,ds=\int\left(u\frac{\partial}{\partial r}F-F\frac{\partial}{\partial n}u\right)dl. \tag{2.4}$$

The right side of (2.4) is equal to zero, since F and its normal derivative at the boundary are zero. Thus, Eq. (2.3) changes to the form

$$\int\int F\Delta u\,ds=\int\int u\,(\sigma_{zz}-\chi)\,ds. \tag{2.5}$$

Equation (2.5) should be satisfied for any u. If u is a harmonic function, then the left side of (2.5) will be equal to zero. Thus, $\sigma_{zz}-\chi$ should be orthogonal to any harmonic function, i.e., χ should be equal to the harmonic part of σ_{zz}.

We will prove that the above-cited condition for χ is sufficient. To do this, we replace u in (2.5) by the elementary solution of the Laplace equation u_1: $u=u_1=[\ln\sqrt{(x-x_0)^2+(y-y_0)^2}]/(2\pi)$. Considering that the Laplace operator of u_1 is equal to the Dirac delta function, we have

$$F(x_0,y_0)=\int\int(\sigma_{zz}-\chi)\ln\sqrt{(x-x_0)^2+(y-y_0)^2}\,dx\,dy/(2\pi).$$

It follows from the necessary properties of χ that u_1 is determined to within an additive harmonic function.

In particular, if the distribution of σ_{zz} depends only on the radius, then the harmonic part of σ_{zz} is equal to a constant, the mean value of σ_{zz} over the cross section. It follows from the equilibrium condition that this mean value is equal to zero, i.e., that χ is also equal to zero. Thus, in this case Eq. (2.2) leads to the summation law [4, 6, 7] $\sigma_{xx}+\sigma_{yy}=\sigma_{zz}$. It is evident from this example that satisfaction of this law depends on the distribution of σ_{zz} but is independent of the form of the cross section.

3. Equation (1.3) has an explicit solution only for particular forms of $\mu(x, y)$. For example, the solution of the boundary-value problem relative to F when $\mu(x, y) = \mu_0/(a + bx + cy)$ nearly reduces to the case μ = const. This result becomes obvious if Eq. (1.3) is changed to the form

$$\Delta\left[\frac{1}{\mu}\Delta F\right] - \Delta\left[\frac{1}{\mu}\sigma_{zz}\right] = \left(\frac{\partial^2}{\partial y^2}F\right)\left(\frac{\partial^2}{\partial x^2}\frac{1}{\mu}\right) + \left(\frac{\partial^2}{\partial x^2}F\right)\left(\frac{\partial^2}{\partial y^2}\frac{1}{\mu}\right) - 2\left(\frac{\partial^2}{\partial x\,\partial y}F\right)\left(\frac{\partial^2}{\partial x\,\partial y}\frac{1}{\mu}\right).$$

With an arbitrary function $\mu(x, y)$, numerical methods must be used to solve the boundary-value problem. Here, it is customary to employ a variational formulation. The solution of the given boundary-value problem is equivalent to finding the function F which satisfies the boundary conditions and gives the extremum of the functional

$$J = \iint \frac{1}{\mu}\left[\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}\right)^2 - \frac{1}{2}\left(\frac{\partial^2 F}{\partial y^2}\right)^2 - \sigma_{zz}\Delta F\right] ds.$$

Without going into the details of different numerical methods, let us examine the dependence of the stress on the shear modulus μ in the example of the axisymmetric problem for a circular two-layer cylinder

$$\mu(r) = \begin{cases} \mu_0 & \text{at} \quad 0 \leqslant r < r_0, \\ \mu_1 & \text{at} \quad r_0 \leqslant r \leqslant 1. \end{cases}$$

The solution will be presented in the cylindrical coordinate system r, φ. We will attempt to find the unknown components $\sigma_{rr} = \sigma_r$, $\sigma_{\varphi\varphi} = \sigma_\varphi$ from the equation of equilibrium

$$\frac{d}{dr}(r\sigma_r) = \sigma_\varphi$$

and the compatibility equation

$$\frac{d}{dr}[(\sigma_\varphi - \sigma_{zz})/\mu] = (\sigma_r - \sigma_\varphi)/\mu r. \tag{3.1}$$

It is difficult to use Eq. (1.3) directly in this case, since μ is a nondifferentiable function. Excluding σ_φ from (3.1), we express σ_{zz} in terms of σ_r:

$$\sigma_{zz} = \frac{d}{dr}(r\sigma_r) - \mu\int_r^1 \frac{1}{\mu}\left(\frac{d}{dx}\sigma_r(x)\right)dx - A.$$

The constant A is determined from the equilibrium condition

$$\int_0^1 r\sigma_{zz}\,dr = 0.$$

After performing some elementary transformations, we obtain

$$\sigma_{zz}(r) = \frac{1}{r}\frac{d}{dr}(r^2\sigma_r) + \frac{\mu_0 - \mu_1}{\mu_1}\sigma_r(r_0)\cdot\begin{cases}(1 - r_0^2) & \text{at} \quad 0 \leqslant r < r_0, \\ (-r_0^2) & \text{at} \quad r_0 \leqslant r \leqslant 1.\end{cases} \tag{3.2}$$

Finding σ_r from the given σ_{zz} by using Eq. (3.2) reduces to integration. The subsequent solution of the problem is elementary.

It should be noted that (3.2) can be written in the form of a modified summation law

$$\sigma_{zz}=\sigma_{\varphi}+\sigma_r+\frac{\mu_0-\mu_1}{\mu_1}\sigma_r(r_0)\cdot\begin{cases}(1-r_0^2) & \text{at} \quad 0\leqslant r<r_0,\\ (-r_0^2) & \text{at} \quad r_0\leqslant r\leqslant 1.\end{cases} \tag{3.3}$$

Apart from their direct use, Eqs. (3.2) and (3.3) make it possible to evaluate the potential of a model with a constant shear modulus μ for multilayered structures.

Returning to the above-examined case, we note that quenching stresses are determined by the method of integral photoelasticity in the following sequence. First we determine σ_{zz} using the inversion of the Radon transform. We then use the solution of boundary-value problem (1.3) to find F, and we use (1.2) to find the remaining stress components.

We thank Kh. K. Aben for his help in completing the study and his participation in the discussion of its results.

LITERATURE CITED

1. A. Q. Tool, "Relation between inelastic deformability and thermal expansion of glass in its annealing range," J. Am. Ceram. Soc., 29, No. 9 (1946).
2. G. M. Bartenev, Mechanical Properties and Heat Treatment of Glass [in Russian], Gosstroiizdat, Moscow (1960).
3. V. L. Indenbom and L. I. Vilro, "Thermoplastic and structural stresses in solids," Fiz. Tverd. Tela, 6, No. 4 (1964).
4. Kh. K. Aben, Integral Photoelasticity [in Russian], Valgus, Tallinn (1975).
5. P. L. Chu and T. Whitbread, "Measurement of stresses in optical fiber and preform," Appl. Opt., 25, No. 7 (1986).
6. H. Poritsky, "Analysis of thermal stresses in scaled cylinders and the effect of viscous flow during anneal," Phys., No. 12 (1934).
7. R. C. O'Rourkl, "Three-dimensional photoelasticity," J. Appl. Phys., 22, No. 7 (1951).
8. H. Aben, "Tomographic optique des champs de constraintes," Rev. Fr. Mecanique, No. 1 (1989).
9. S. P. Timoshenko and J. N. Goodier, Theory of Elasticity, McGraw-Hill, New York (1969).
10. T. Abe, Y. Misunage, and H. Koga, "Novel measurement method for axial residual stress in optical fibre," Electron. Lett., 21, No. 1 (1985).
11. A. É. Puro, "Tomography with weak optical anisotropy," Summary of Documents of the Fourth All-Union Symposium on Computer Tomography, Vol. 1, IM SO AN SSSR, Novosibirsk (1989).
12. R. M. Lewitt, "Reconstruction algorithms: transform methods," Proc. IEEE, 71, No. 3 (1983).
13. V. Novatskii, Theory of Elasticity [Russian translation], Mir, Moscow (1975).
14. L. Levi, Applied Optics: A Guide to Optical System Design, Vol. 2, New York (1980).